B.Sc. part II (H), paper III Curvature. 29.07.2020

Q.1 For the curve $r = a(1+\cos\theta)$ (polar curve), prove that $\frac{\rho^2}{r}$ is constant.

Soln: Here, $r_1 = -a\sin\theta$, $r_2 = -a\cos\theta$

$$\therefore \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]^{\frac{3}{2}}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2\cos\theta(1+\cos\theta)}$$

$$= \frac{a^3(2+2\cos\theta)^{\frac{3}{2}}}{a^2[3+3\cos\theta]} = \frac{2^{3/2}}{3}a(1+\cos\theta)^{\frac{1}{2}} = \frac{2^{3/2}}{3}a^{\frac{1}{2}}\sqrt{r}$$

$$\therefore \frac{\rho^2}{r} = \frac{8a}{9} = \text{constant.}$$

Q.2. Show that the radius of curvature of the cardioide $r = a(1+\cos\theta)$ varies as $\sqrt{r}$

Soln: Let $r_1 = -a\sin\theta$, $r_2 = -a\cos\theta$

$$\therefore \rho = \frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2} = \frac{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]^{\frac{3}{2}}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2\cos\theta(1+\cos\theta)}$$

$$= \frac{a^3(2+2\cos\theta)^{\frac{3}{2}}}{a^2[3+3\cos\theta]} = \frac{2^{\frac{3}{2}}}{3}a(1+\cos\theta)^{\frac{1}{2}} = \frac{2^{\frac{3}{2}}}{3}a^{\frac{1}{2}}\sqrt{r}.$$

$$\rho = \frac{2^{\frac{3}{2}}}{3}a^{\frac{1}{2}}\sqrt{r} = (\text{constant})\sqrt{r}$$

$\therefore \rho \propto \sqrt{r}$ Proved.

Q.3. For the curve $r^m = a^m\cos m\theta$, prove that
$$\rho = \frac{a^m}{(m+1)r^{m-1}}$$

Soln: Now, $r^m = a^m\cos m\theta$

Taking logarithm of both sides, we get

$$m\log r = m\log a + \log\cos m\theta$$

Diff. both sides w.r.to θ,

$$m\cdot\frac{1}{r}\frac{dr}{d\theta} = -m\tan m\theta$$

$$\cot\phi = \cot\left(\frac{\pi}{2}+m\theta\right)$$

$$\therefore \phi = \frac{\pi}{2}+m\theta$$

Again, $p = r\sin\phi = r\sin\left(\frac{\pi}{2}+m\theta\right) = r\cos m\theta$

or, $p = r\,\frac{r^m}{a^m}$

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proof Q 3.

$$a^m p = r^{m+1}$$

Diff. both sides w.r.to r,

$$a^m \frac{dp}{dr} = (m+1) r^m$$

$$\therefore \rho = r\frac{dr}{dp} = r\frac{a^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}$$

Q.4. For any curve prove that the formula

$$\rho = \frac{r}{\sin\phi\left(1+\frac{d\phi}{d\theta}\right)}, \text{ where } \tan\phi = r\frac{d\theta}{dr}$$

Soln: We know that $\sin\phi = r\frac{d\theta}{dr}$, $\rho = \frac{ds}{d\psi}$

and $\theta + \phi = \psi$

Now, $\sin\phi\left(1+\frac{d\phi}{d\theta}\right) = r\frac{d\theta}{ds} + r\frac{d\theta}{dr}\cdot\frac{d\phi}{d\theta} = \left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right)r$

$$= r\frac{d}{ds}(\theta+\phi) = r\frac{d\psi}{ds} = \frac{r}{\rho}$$

$$\therefore \rho = \frac{r}{\sin\phi\left(1+\frac{d\phi}{d\theta}\right)}$$ Proved.

Q.5. Find the radius of curvature at any point on the ellipse $p^2(a^2+b^2-r^2) = a^2b^2$

Soln: $\because a^2+b^2-r^2 = b^2a^2p^{-2}$

Diff. w.r.to p, we get

$$-2r\frac{dr}{dp} = -2a^2b^2p^{-3}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{a^2b^2}{p^3}$$

$$\because \rho = r\frac{dr}{dp}$$

$$\therefore \rho = \frac{a^2b^2}{p^3}$$ Proved.

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